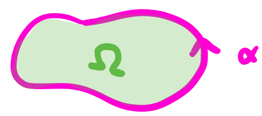
§ Some global theorems for plane curves (continued)

Isoperimetric Inequality

For any simple closed curve in $\mathbb{R}^2$,

$$\boxed{A \leq \frac{L^2}{4\pi}}$$



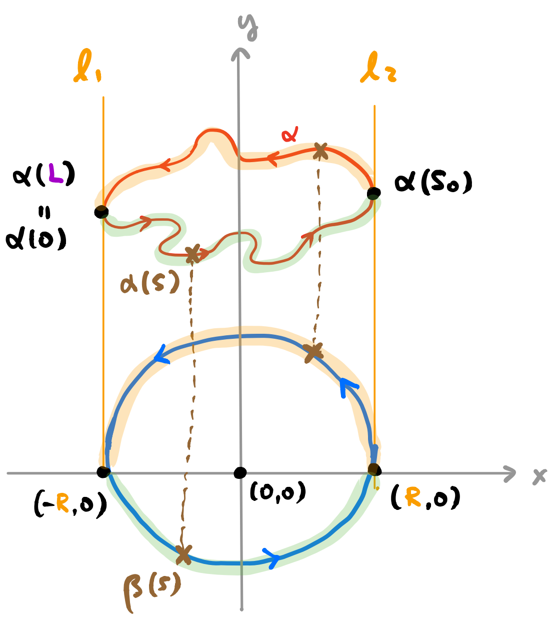
$L = \text{Length}(\alpha)$

$A = \text{Area}(\Omega)$

"=" holds <=> round circle

Proof: Let $\alpha(s) : [0, L] \longrightarrow \mathbb{R}^2$ be a simple closed plane curve p.b.a.l.

Consider two parallel vertical lines ℓ_1, ℓ_2 touching the curve α on both sides, enclose a circle in the same slab like this:



$\alpha(s) = (x(s), y(s))$, $s \in [0, L]$

$\beta(s) = (\bar{x}(s), \bar{y}(s))$, $s \in [0, L]$

where $\bar{x}(s) = x(s)$ and

$\bar{y}(s) = \pm\sqrt{R^2 - x(s)^2}$.

Recall: (Green's Thm)

$$\int_{\partial\Omega} P\,dx + Q\,dy = \iint_{\Omega} \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dx\,dy$$

$$\Rightarrow \text{Area}(\Omega) = \int_{\partial\Omega} x\,dy = -\int_{\partial\Omega} y\,dx$$

Let A and $\bar{A}$ be the area enclosed by α and β respectively.

Using Green's Thm, we compute

$$A = \int_0^L x(s)\, y'(s)\, ds$$

$$\pi R^2 \underset{\substack{\uparrow \\ \beta \text{ is a circle} \\ \text{of radius } R}}{=} \bar{A} = -\int_0^L \bar{y}(s)\, \bar{x}'(s)\, ds \underset{\substack{\uparrow \\ \bar{x}(s) = x(s)}}{=} -\int_0^L \bar{y}(s)\, x'(s)\, ds$$

Adding these up & use A.M.-G.M. inequality (i.e. $\sqrt{ab} \leq \frac{a+b}{2}$):

$$\begin{aligned}
\sqrt{A}\sqrt{\pi R^2} &\leq \frac{1}{2}\left(A + \pi R^2\right) \\
&\leq \frac{1}{2}\int_0^L \left(x(s)\, y'(s) - \bar{y}(s)\, x'(s)\right) ds \\
&= \frac{1}{2}\int_0^L \langle (x(s), \bar{y}(s)), (y'(s), -x'(s)) \rangle\, ds \\
\left(\substack{\text{Cauchy-} \\ \text{Schwarz}}\right) &\leq \frac{1}{2}\int_0^L \underbrace{|(\bar{x}(s), \bar{y}(s))|}_{\equiv} \cdot \underbrace{|(y'(s), -x'(s))|}_{\equiv 1}\, ds \\
&= \frac{1}{2} R L
\end{aligned}$$

Simiplying gives $\quad 4\pi A \leq L^2.$

It remains to analyze the equality case.

Suppose $4\pi A = L^2$. Then, we have all the inequalities above are achieved as equalities. In particular,

$$A = \pi R^2 \quad \text{and} \quad L = 2\pi R$$

where R is <u>independent</u> of the choice of ℓ_1, ℓ_2.

Also, equality case of Cauchy-Schwarz inequality gives

$$\underbrace{(x, \bar{y})}_{\text{length } = R} \,/\!/\, \underbrace{(y', -x')}_{\text{length } = 1}$$

So, $(x, \bar{y}) = R\,(y', -x')$, thus $x = R y'$.

Switching the roles of x & y coordinates, and using the invariance of R, we have also $y - y_0 = R x'$ for some constant y_0. Therefore,

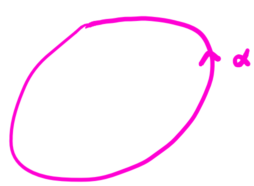
$$\begin{aligned} x(s)^2 + (y(s) - y_0)^2 &= (R\, y'(s))^2 + (R\, x'(s))^2 \\ &= R^2 \underbrace{(x'(s)^2 + y'(s)^2)}_{\equiv 1} = R^2 \end{aligned}$$

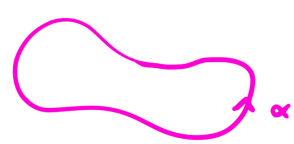
So, α lies on a circle of radius R centered at $(0, y_0)$.

$\square$

<u>Def^{n}</u>: A plane curve is said to be convex if $k \geq 0$ everywhere.

E.g.)



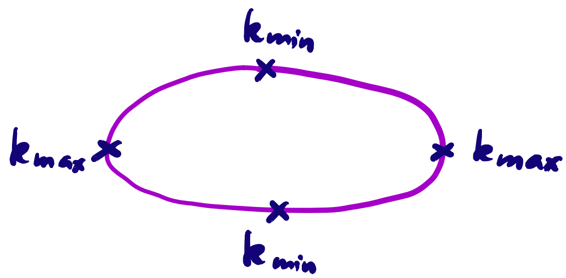


Four Vertex Theorem

For any simple closed (convex) curve in $\mathbb{R}^2$,

$\exists$ at least 4 vertices (i.e. points where $k' = 0$)

ellipse has exactly 4 vertices.



Proof: omitted.

Note: It is easy to show that $\exists$ 2 vertices, where k achieves its maximum and minimum. What is non-trivial is that there are at least 2 more!